Weakening the independence axiom

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Febraury 2020

Recollect the three axioms of Mixture Space Theorem, that choices are given by preference relation, they satisfy the independence axiom and the continuity axiom. While the third is often regarded as a "technical" axiom, the first two have been repeatedly shown to be violated in various experimental settings. Gilboa gives a nice explanation of what to make of violations of axioms in Chapter 12. Specifically, we should ask– when (or in what context) is the theory still useful, even though the axioms it rests on have been shown to be violated in certain settings, and when is it not.

1 Alias' paradox

The most famous violation of the independence axiom is due to Alias, and is often referred to as the Alias' paradox. For a nice exposition, read Section 1.2 and look at Figure 1 and 2 in "The two faces of independence: betweenness and homotheticity" by Burghart, Epper and Ernst Fehr.

One of the fixes to rationalize Alias type behavior is to give up the independence axiom and invoke the betweenness axiom.

2 Dekel's betweenness

Let $X = [\underline{x}, \bar{x}] \subset \mathbb{R}, \underline{x} < \bar{x}$, be the set of prizes. Let $\mathcal{L}$ denote the set of all simple lotteries on $[\underline{x}, \bar{x}]$. Let $\succsim$ be a binary relation on X.

A note on notation. We will use $x, \underline{x}, \bar{x}$ both as members of X and also as degenerate lotteries. So when we are talking about lotteries, x would mean δ_x, etc.

Axiom 1: $\succsim$ is a preference relation, with $\bar{x} \succ \underline{x}$.

Axiom 2 (Solvability): For all $p, q, r \in \mathcal{L}$, $p \succ q \succ r$ implies there exists $a \in (0, 1)$ such that $ap + (1 - a)r \sim q$.

Axiom 3 (Monotonicity): For all $x, y \in X$, $p \in \mathcal{L}$, and $a \in (0, 1)$, $x \succ y$ implies $ap + (1 - a)x \succ ap + (1 - a)y$.

Axiom 4 (Betweenness): For all $p, q \in \mathcal{L}$ and $a \in (0, 1)$, $p \succ q$ implies $p \succ ap + (1 - a)q \succ q$, and $p \sim q$ implies $p \sim ap + (1 - a)q$.

Lemma 1. *$p \succ q$ and $0 \le a < b \le 1 \implies bp + (1-b)q \succ ap + (1-a)q$.*

Proof. If $a = 0$ or $b = 1$, then the result follows from Axiom 4. So, let $0 < a, b < 1$. Now, by Axiom 4, we know that $p \succ ap + (1-a)q$. For any $c \in (0,1)$ consider the lottery $cp + (1-c)[ap + (1-a)q] = [c + (1-c)a]p + [(1-c)(1-a)]q$. Using Axiom 4 again, we know that $p \succ [c + (1-c)a]p + [(1-c)(1-a)]q \succ ap + (1-a)q$. Finally we want to choose c such that $c + (1-c)a = b$, that is, $c = \frac{b-a}{1-a}$. Clearly, this c lies in $(0,1)$ and by the choice of c, we get $bp + (1-b)q \succ ap + (1-a)q$. □

Lemma 2. *If $x > y$, then $x \succ y$.*

Proof. Let $x > y$. Fix $a \in (0,1)$. Then by Axiom 3,

$$\begin{aligned} x = ax + (1-a)x &\succ ax + (1-a)y \\ &= (1-a)y + ax \\ &\succ (1-a) + a(y) = y. \end{aligned}$$

□

Theorem 1. *$\succsim$ satisfies Axioms 1-4 iff there exist functions $u : X \times [0,1] \to \mathbb{R}$ and $V : \mathcal{L} \to [0,1]$ such that V is onto, u is continuous in its second argument, strictly increasing in the first argument, V represents $\succsim$,*

$$V(p) = \sum_x u(x, V(p))p(x)$$

and $p \succsim q$ iff

$$\begin{aligned} \sum_x u(x, V(p))p(x) &\ge \sum_x u(x, V(p))q(x), \text{ and} \\ \sum_x u(x, V(q))p(x) &\ge \sum_x u(x, V(q))q(x) \end{aligned}$$

Proof. We break the proof into logical steps.
Step 1. Construct V that represents $\succsim$. For all $p \in \mathcal{L}$, we know by Axiom 2 that there exists $a \in [0,1]$ such that $p \sim a\bar{x} + (1-a)\underline{x}$. define $V(p) = a$.We claim that V is well defined and represents $\succsim$. Suppose $p \sim a\bar{x} + (1-a)\underline{x}$ and $p \sim b\bar{x} + (1-b)\underline{x}$, where $a, b \in [0,1]$. Then, $a\bar{x} + (1-a)\underline{x} \sim b\bar{x} + (1-b)\underline{x}$. Using lemma 1, it is clear that $a = b$. Thus, V is well defined.

Next, we show that V represents $\succsim$. Let $p, q \in \mathcal{L}$. Suppose $V(p) = V(q) = a$. Let $p \sim a\bar{x} + (1-a)\underline{x} \sim q$. Next, suppose $V(p) > V(q)$. Let $V(p) = a$ and $V(q) = b$. We want to show that $p \succ q$. Suppose $q \succsim p$. If $q \sim p$, then by definition of V, $b\bar{x} + (1-b)\underline{x} \sim a\bar{x} + (1-a)\underline{x}$. Again using lemma 1, we get $a = b$ which is a contradiction. If $q \succ p$, then we have $b\bar{x} + (1-b)\underline{x} \succ a\bar{x} + (1-a)\underline{x}$. lemma 1 implies $b > a$, contradiction. Thus, we must have $p \succ q$.

Thus, we have shown $V(p) = V(q)$ implies $p \succ q$ and $V(p) > V(q)$ implies $p \succ q$. This equivalently shows that V represents $\succsim$.

Step 2. Constructing u(.,.). Now for any degenerate lottery $x \in \mathcal{L}/\{\underline{x}, \bar{x}\}$ and $a \in (0,1)$, only one of the following holds:*(i)* $x \sim a\bar{x} + (1-a)\underline{x}$, *(ii)* $x \succ a\bar{x} + (1-a)\underline{x}$, or *(iii)* $a\bar{x} + (1-a)\underline{x} \succ x$. For each of the cases define $u(x,a)$ in the following way.

(i) $x \sim a\bar{x} + (1-a)\underline{x}$. Let $u(x,a) = a$.

(ii) $x \succ a\bar{x} + (1-a)\underline{x} \succ \underline{x}$. Then, we know by Axiom 2 that there exists $b \in (0,1]$ such that $b\bar{x} + (1-b)\underline{x} \sim a\bar{x} + (1-a)\underline{x}$. Define $u(x,a) = \frac{a}{b}$.

(iii) $\bar{x} \succ a\bar{x} + (1-a)\underline{x} \succ x$. Then, we know by Axiom 2 that there exists $b \in [0,1)$ such that $b\bar{x} + (1-b)\underline{x} \sim a\bar{x} + (1-a)\underline{x}$. Define $u(x,a) = \frac{a-b}{1-b}$.[1]

Further, define $u(\bar{x},a) = 1$ and $i = u(\underline{x}) = 0$ for all $a \in (0,1)$. Finally, what about $u(x,a)$ when $a \in \{0,1\}$? Since u will be shown to be continuous in the second argument in the interval $(0,1)$, we extend the definition of $u(x,a)$ to the closed interval by continuity.

Step 3. u(x,.) is continuous on $(0,1)$ *and* $u(.,a)$ *is strictly increasing on* X. First continuity. Fix $x \in X$. If $x \in \{\bar{x}, \underline{x}\}$, then $u(x,.)$ is a constant function, and hence continuous on $(0,1)$. So, let $x \in X/\{\bar{x}, \underline{x}\}$. Define $B(x) = \{a | a\bar{x} + (1-a)\underline{x} \sim x\}$. By Axiom 2, we know the set is nonempty. Note that Axiom 4 and lemma 1 $\implies$ $B(w) = [\bar{a}, 1)$ for some $\bar{a}$. Next, as we did in the construction of u, for a fixed $a \in (0,1]$, there exists $b \in (0,1]$ such that $b\bar{x} + (1-b)\underline{x} \sim a\bar{x} + (1-a)\underline{x}$. This implicitly defines a function $b(a)$. Clearly, $b(\bar{a}) = 0$ and $b(1) = 1$. Also, staring at lemma 1 a little bit, it is clear that $b(.)$ should be increasing. Now, we show that $b(.)$ is continuous. If not, then there exists a sequence $a_n \uparrow a$ in $B(x)$, such that $b(a) > lim\ b(a_n)$. So, for $\hat{b}$ satisfying $b(a) > \hat{b} > b(a_n)$, it is clear that $a\bar{x} + (1-a)\underline{x} \succ \hat{b}\bar{x} + (1-\hat{b})\underline{x} \succ a_n\bar{x} + (1-a_n)\underline{x}$ for every $n \in \mathbb{N}$. Hence, there exists a (unique) $\hat{a}$ such that $\hat{b}\bar{x} + (1-\hat{b})\underline{x} \sim \hat{a}\bar{x} + (1-\hat{a})\underline{x}$ and $\hat{a} \in (a_n, a)$ for every n. But this cannot be true since $a_n \uparrow a$. So, we must have $b(.)$ continuous. Now, recall that by construction of u, we know that $u(x,a) = \frac{a-b(a)}{1-b(a)}$ for $a \in (\bar{a}, a)$, and $u(x,\bar{a}) = \bar{a}$. So, clearly $u(x,.)$ is continuous on $[\bar{x}, 1)$. Similarly, we can show that it is continuous on $(0, \bar{a}]$.

It is easy to check that u is strictly increasing in the first argument. Fix $a \in [0,1]$ and let $x \succ y$ (and thus $x \succ y$). Let $p = a\bar{x} + (1-a)\underline{x}$. If $x \succ p \succ y$, then by construction $u(x,a) > a > u(y,a)$. Consider the case when $x \succ y \succ p$. Then, by construction $u(x,a) = \frac{a}{b}$ where $bx + (1-b)\underline{x} \sim p$, and $u(x,a) = \frac{a}{b'}$ where $b'y + (1-b')\underline{x} \sim p$. We claim that $b' > b$. Suppose $b' = b$. Then by Axiom 5, $(1-b)\underline{x} + bx \succ (1-b')\underline{x} + b'y$, which is a contradiction. Suppose $b' > b$. Then,

$$
\begin{aligned}
bx + (1-b)\underline{x} &= (1-b)\underline{x} + bx \\
&\succ (1-b)\underline{x} + by && \text{(by Axiom 3)} \\
&= by + (1-b)\underline{x} \\
&\succ b'y + (1-b')\underline{x} && \text{(by lemma 1)}
\end{aligned}
$$

[1]These definitions are not randomly being pulled out of a hat. Suppose a linear U that represents $\succsim$ and $U(\bar{x}) = 1$, $U(\underline{x}) = 0$. Then, in case *(ii)*, $bx + (1-b)\underline{x} \sim a\bar{x} + (1-a)\underline{x}$ would imply $U(bx + (1-b)\underline{x}) = U(a\bar{x} + (1-a)\underline{x})$, that is $bU(x) + (1-b)U(\underline{x}) = aU(\bar{x}) + (1-a)U(\underline{x})$. Thus, $bU(x) + 0 = a + 0$, which implies $U(x) = \frac{a}{b}$. Similarly for *(iii)*.

which is a contradiction. Thus we must have $b' > b$, which implies $\frac{a}{b} > \frac{a}{b'}$, that is, $u(x, a) > u(x, b)$. The case $p > x > y$ can be similarly done.

Step 4. Bridge u and V. Define $U(p, a) = \sum_x u(x, a)p(x)$. We aim to show that

$$U(p, V(p)) = V(p)$$

Suppose $p = a\bar{x} + (1 - a)\underline{x}$ for some $a \in [0, 1]$. Then, $V(p) = a$ and $U(p, V(p)) = U(p, a) = au(\bar{x}, a) + (1 - a)u(\underline{x}, a) = aV(p)$.

Next, suppose $p = x$ for some $x \in X$. If $p = \bar{x}$ then, $U(p, V(p)) = u(\bar{x}, V(\bar{x})) = u(\bar{x}, 1) = 1 = V(p)$. Similarly for $p = \underline{x}$. Now let $p = x$, where $x \in (\underline{x}, \bar{x})$ Then, $U(p, V(p)) = u(x, V(p))$ and $p = x \sim V(p)\bar{x} + (1 - V(p))\underline{x}$. We know by lemma 2 that $\underline{x} < x < \bar{x}$ implies $\bar{x} > x > \underline{x}$. Thus, $V(p) \in (0, 1)$. So, we are in part *(i)* of the definition of u. Therefore, $U(p, V(p)) = u(x, V(p)) = V(p)$, as desired.

Now, suppose $p = bx + (1 - b)\underline{x}$, where $b \in (0, 1]$ and $x \in (\underline{x}, \bar{x})$. Then, by definition, $V(p)\bar{x} + (1 - V(p))\underline{x} \sim p$. Thus we have $bx + (1 - b)\underline{x} \sim V(p)\bar{x} + (1 - V(p))\underline{x} \sim p$. Therefore, by construction, $u(x, V(p)) = \frac{V(p)}{b}$. Hence, we get $U(p, V(p)) = bu(x, V(p)) + (1 - b)u(\underline{x}, V(p)) = b\frac{V(p)}{b} = V(p)$. The case where $p = b\bar{x} + (1 - b)x,\ b \in [0, 1)$ can be similarly done.

For any $p \in \mathcal{L}$, let

$$\tilde{S}_p = \{x \in X | x \notin \{\underline{x}, \bar{x}\} \text{ and } p(x) > 0\}, \text{ and } k(p) = |\tilde{S}_p|.$$

To complete the argument for this step we will use induction on $k(p)$. If $k(p) = 0$, then $p = a\bar{x}+(1-a)\underline{x}$ for some $a \in [0, 1]$, and we already covered this case. Let $n \geq 0$, and $U(p, V(p)) = V(p)$ for all p such that $k(p) = n$. Now let $k(p) = n + 1$, thus $k(p) \geq 1$. We want to show that $U(p, V(p)) = V(p)$. Pick $x \in \tilde{S}_p$. Then, $p = cx + (1 - c)q$ for some $c \in (0, 1)$ and $q \in \mathcal{L}$ such that $k(q) = n$.

If $x \sim p$, then we claim that $q \sim p$. Suppose $p > q$. Then, $x > q$. By Axiom 4, $x > cx + (1 - c)q = p$, a contradiction. Suppose $q > p$. Then, $q > x$. By Axiom 4, $p = cx + (1 - c)q > x$, contradiction. Thus, we must have $q \sim p$. Note that $V(q) = V(p) = V(x)$ and by inductive hypothesis, $U(q, V(q)) = V(q)$. Thus, $U(p, V(p)) = cu(x, V(p)) + (1 - c)U(q, V(p)) = cu(x, V(x)) + (1 - c)V(q) = cV(x) + (1 - c)V(q) = V(p)$.

Now consider the case where $x > p$. Let $\hat{b} = p(x) > 0$. Then, we know that $p = \hat{b}x + (1 - \hat{b})q$ where $q \in \mathcal{L}$ such that $k(q) = n$. Now, $\bar{x} > x > p > q > \underline{x}$. ($q = \{\bar{x}\}$ is not possible and $q = \{\underline{x}\}$ has already been covered as special case). By Axiom 2, choose $b_1, b_2 \in (0, 1)$ such that

$$p_1 = b_1x + (1 - b_1)\underline{x} \sim p, \text{ and}$$
$$p_2 = b_2\bar{x} + (1 - b_2)q \sim p$$

Then, $V(p_1) = V(p_2) = V(p)$. Further, $U(p_1, V(p)) = U(p_1, V(p_1)) = V(p_1)$ by the earlier special case. Also, since $k(p_2) = n$, by the induction hypothesis $U(p_2, V(p)) = U(p_2, V(p_2)) = V(p_2)$. Thus, $U(p_1, V(p_1)) = U(p_2, V(p_2)) = V(p)$. Let $V(p) = a$.

It is possible to find c_1 and $c_2 \in [0,1]$ such that

$$\begin{aligned}
c_1 p_1 + (1-c_1)p_2 &= c_1[b_1 x + (1-b_1)\underline{x}] + (1-c_2)[b_2 \bar{x} + (1-b_2)q] \\
&= c_2[a\bar{x} + (1-a)\underline{x}] + (1-c_2)[\hat{b}x + (1-\hat{b})q] \\
&= c_2[a\bar{x} + (1-a)\underline{x}] + (1-c_2)p.
\end{aligned}$$

Since$p \sim p_1 \sim p_2 \sim c_1p_1 + (1-c_1)p_2$ (Axiom 4) we have $a = V(p) = V(p_1) = V(p_2) = V(c_1p_1 + (1-c_1)p_2)$. Thus, we get,

$$\begin{aligned}
U(c_1p_1 + (1-c_1)p_2, V(c_1p_1 + (1-c_1)p_2)) &= c_1U(p_1, V(p_1)) + (1-c_1)U(p_2, V(p_2)) \\
&= c_1V(p_1) + (1-c_1)V(p_2) \\
&= V(p).
\end{aligned}$$

Thus,

$$\begin{aligned}
V(p) &= U(c_2[a\bar{x} + (1-a)\underline{x}] + (1-c_2)p, a) \\
&= c_2U(a\bar{x} + (1-a)\underline{x}, a) + (1-c_2)U(p, a) \\
&= c_2V(p) + (1-c_2)U(p, a).
\end{aligned}$$

This implies, $U(p,a) = V(p)$, that is, $U(p, V(p)) = V(p)$. Therefore, $U(p, V(p)) = V(p)$ for all $p \in \mathcal{L}$, as desired.

Step 5. *$p \succsim q$ iff the two inequalities in the statement hold.* If $p \sim q$, then $V(q) = V(p)$. Thus, $U(p, V(p)) = U(q, V(q)) = U(p, V(q)) = U(q, V(p))$ and the inequalities hold trivially. Suppose $p \succ q$. The case where $p = \bar{x}$ or $q = \underline{x}$ are straightforward so we look at the case where $\bar{x} \succ p \succ q \succ \underline{x}$. Then, there exists $b \in (0,1)$ such that $b\bar{x} + (1-b)q \sim p$. This implies $U(p, V(p)) = U(b\bar{x} + (1-b)q, V(p)) = b + (1-b)U(q, V(p)) \geq U(q, V(p)) \geq U(q, V(p))$, with equality iff $U(q, V(p)) = 1$. Again by Axiom 2, there exists $c \in (0,1)$ such that $cp + (1-c)\underline{x} \sim q$. This implies $U(q, V(q)) = U(cp + (1-c)\underline{x}, q) = cU(p, V(q)) \leq U(p, V(q))$, with equality iff $U(p, V(q)) = 0$.

Now, $U(q, V(p)) = 1$ is possible only if $p = \bar{x}$ which we have ruled out. Similarly, we cannot have $U(p, V(q)) = 0$. Thus, we have established that $p \succ q \implies U(p, V(p)) > U(q, V(p))$ and $U(p, V(q)) > U(q, V(q))$. Conversely suppose we have $U(p, V(p)) \geq U(q, V(p))$ and $U(p, V(q)) \geq U(q, V(q))$. If $q \succ p$, then by the above arguments we will have $U(q, V(p)) > U(p, V(p))$ and $U(q, V(q)) > U(p, V(q))$, which is a contradiction. Thus, we have $p \succsim q$, and we have proved the second part of the theorem as well. □

Dekel argues for the suitability of the representation in many ways, one of which is the relationship between concavity and risk aversion. Say that $\succsim$ on $\mathcal{L}$ is risk averse if

Proposition 1. *Let u and V represent $\succsim$ as described in Theorem 1. Then, $\succsim$ is risk-averse iff u in the first argument.*

Proof. $\square$

3 Gul's disappointment aversion