Savage's theory of choice under uncertianty

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Kreps says "Savage's theory, which is the crowning glory of choice theory,.." Fishburn states "The most brilliant axiomatic theory of utility ever developed is, in my opinion, the expected utility theory of Savage." Before starting the class on Savage, Faruk Gul looked at us in zen state and exclaimed, "This is a big lecture for us, we will learn Savage"

0 Preliminaries, axioms and representation theorem

Let X be an arbitrary set of prizes and Ω be an arbitrary set of states. A Savage act is a function $f : \Omega \to X$. Let F be the set of all Savage acts (given X and Ω). Let $\mathcal{A}$ be the algebra of all subsets of Ω. A function $\mu : \mathcal{A} \to \mathbb{R}$ is called a probability measure if it satisfies the following condition: (i) $\mu(\Omega) = 1$, (ii) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$, and (iii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$.

Let $\succsim$ be a binary relation on F. We write $x \in F$ to denote the constant act that always yields x. We state the Savage axioms:

Axiom 1: $\succsim$ is a preference relation.

Axiom 2: $\exists x, y \in X$ such that $x \succ y$.

Let $A \subset \Omega$. Define fAg to be the act h where,

$$h(s) = \begin{cases} f(s) & \text{if } s \in A \\ g(s) & \text{if } s \notin A \end{cases}$$

Definition 1. *Let $A \subset \Omega$. Then, A is said to be null if $fAh \sim gAh$ for all $f, g, h \in F$.*

Back to the axioms.

Axiom 3: For each $h \in H$, non-null A and $x, y \in X$, $x \succsim y$ iff $xAh \succsim yAh$.

Axiom 4: Let $f, g, h \in F$. Then, $fAh \succsim gAh \implies fAh' \succsim gAh'$ for all $h' \in F$.[1]

[1]Note that Axiom 4 and Axiom 1 imply the following version also exists

$$\forall f, g, h \in F,\ fAh \succsim gAh \implies fAh' \succ gAh'\ \forall h' \in F.$$

Given Axiom 4, we can define conditional preference. We say that $f \succsim g$ given A if $fAh \succsim gAh$ for all $h \in F$. Note that A is null iff $f \succ g$ given A for all $f, g \in F$.

Axiom 5: Let x, y, x', y' be such that $x \succ y$ and $x' \succ y'$. Then for all $A, B \subset \Omega$, $xAy \succsim xBy \implies x'Ay' \succsim x'By'$.

Axiom 6: Let any $x \in X$ and $f, g \in F$ such that $f \succ g$. then $\exists$ a finite partition, $\{A_1, A_2, \ldots, A_n\}$, of Ω such that $xA_if \succ g$ and $f \succ xA_ig$ for all $i = 1, 2, \ldots, n$.

Let $F_0 = \{f \in F \,||f(\Omega)| < \infty\} \subset F$. That is, we define F_0 to be the set of all simple acts on Ω. Another way to think about this restriction is that we take the set of prizes X to be a finite set.

We now set on a long and exciting journey below with an eventual aim of proving the following main result.

Theorem 1. *(Savage). Let $\succsim$ be a binary relation on F_0. Then, $\succsim$ satisfies Axioms 1-6 iff $\exists$ a non-constant function $u : X \to \mathbb{R}$ and a probability measure $\mu : \mathcal{A} \to [0, 1]$ on $(S, \mathcal{A})$ such that the function $W : F \to \mathbb{R}$ defined by*

$$W(f) = \sum_{x \in X} u(x)\mu(f^{-1}(x))$$

represents $\succsim$. Furthermore if W' defined by $W'(f) = \sum_{x \in X} u'(x)\mu'(f^{-1}(x))$ also represents $\succsim$, then $u' = a + bu$ for some $b > 0$ and $\mu' = \mu$.

The extension of the result from F_0 to F requires an additional axiom.

Axiom 7: Let any $f, g \in F$ and any $A \subset \Omega$. Then

$$f \succsim g(s) \text{ given A } \forall s \in A \Rightarrow f \succsim \text{ given A, and}$$

$$f(s) \succsim g \text{ given A } \forall s \in A \Rightarrow f \succsim g \text{ given A.}$$

Theorem 2. *(Generalization to arbitrary acts). Let $\succsim$ be a binary relation on F. Then, $\succsim$ satisfies Axioms 1-7 iff $\exists$ a non-constant function $u : X \to \mathbb{R}$ and a probability measure $\mu : \mathcal{A} \to [0, 1]$ on $(S, \mathcal{A})$ such that the function $W : F \to \mathbb{R}$ defined by*

$$W(f) = \int_{x \in X} u(x)\mu(f^{-1}(x))dx$$

represents $\succsim$. Furthermore if W' defined by $W'(f) = \int_{x \in X} u'(x)\mu'(f^{-1}(x))$ also represents $\succsim$, then $u' = a + bu$ for some $b > 0$ and $\mu' = \mu$.

In what follows, we will proof Theorem 1, and the proof is divided into four parts, split into the next four sections.

1 Qualitative probability

Suppose $A, B \in \mathcal{A}$. then, what does it mean to say

> "I *believe* that event A is more likely than event B."

What does the DM mean by 'more likely'? Since choice theory is all about dealing in probabilities, how about an isomorphism between 'more likely' and 'more probable'? An intuitive way would be the following. Define a binary relation $\succsim^*$ on $\mathcal{A}$, so that $A \succsim^* B$ represents A being more likely then B. We want to find a probability measure $\mu : \mathcal{A} \to [0,1]$ such that

$$A \succsim^* B \text{ iff } \mu(A) \geq \mu(B)$$

This section is devoted to the search of such a nice binary relation and probability measure. Let's make things formal.

Definition 2. *A binary relation $\succsim^*$ on $\mathcal{A}$ is called a qualitative probability if*

(a) $\succsim^*$ *is a preference relation,*

(b) $A \succsim^* \emptyset \ \forall A \in \mathcal{A}$,

(c) $\Omega \succ^* \emptyset$, *and*

(d) $\forall A, B, C \in \mathcal{A}$, $[A \cap C = B \cap C = \emptyset] \implies [A \succ^* B$ *iff* $A \cup B \succ^* B \cup C]$.

Moreover $\succsim^*$ is said to be *divisible* if for all $A, B \in \mathcal{A}$ such that $A \succ^* B$, there exists a finite partition $\{C_1, C_2, \dots, C_n\}$ of Ω such that $A \succ^* B \cup C_i$, for all $i = 1, 2, \dots, n$.

The next obvious question is, given a binary relation $\succsim$ on F that satisfies the axioms stated above, can we find a qualitative probability on $\mathcal{A}$? Or better still, can we find one that is divisible?

Yes of course! We do precisely that in the next theorem.

Definition 3. *For any $A, B \in \mathcal{A}$, we say that $A \succsim^* B$ if for all $x, y \in X$ such that $x \succ y$, we have that $xAy \succsim xBy$. also, $A \succ^* B$ and $A \sim^* B$ are defined in the usual way as $\neg B \succsim^* A$ and $[A \succsim^* B$ and $B \succsim^* A]$, respectively.*

Note that Axiom 2 ensures that there exists $x, y \in X$ such that $x \succ y$ and by Axiom 5 it does not matter which x and y are used, that is, in the definition we could have used "for some x and y" instead of "for all x and y".

Also note that $A \succ^* B$ implies $xAy \succ xBy$. This is because $A \succ^* B$ implies $xAy \succsim xBy$. Suppose $xBy \succsim xAy$. Then since $x \succ y$, by definition, $B \succsim^* A$, which is a contradiction. Therefore, $xAy \succsim xBy$ and $\neg xBy \succsim xAy$, implying $xAy \succ xBy$.

From now on, whenever we talk about $\succsim^*$, we mean the above defined binary relation on $\mathcal{A}$.

Lemma 1. *Suppose Axioms 1-5 hold. Then $\succsim^*$ is a qualitative probability on $\mathcal{A}$. If in addition Axiom 6 holds, then $\succsim^*$ is also divisible.*

Proof. □

Remark 1. *Aslo, note that $A \in \mathcal{A}$ is null iff $A \sim^* \emptyset$. Suppose A is null. Then, by definition $xAy \sim yAy$. This implies $xAy \sim y\emptyset y$. Thus, by definition $A \sim^* \emptyset$. Conversely, let $A \sim^* \emptyset$. Suppose A is not null. Then, by Axiom 3 we have $xAy \succ yAy$. This implies $xAy \succ y\emptyset y$, thus $A \succ^* \emptyset$ by definition, contradiction.*

Now that we have the desired nice binary relation on the set of events, our next step is obviously to manufacture the probability measure 'isomorphic' to it. The next theorem does it for us.

Proposition 1. *Suppose Axioms 1-6 hold. Then there exists a unique probability measure μ on $(\Omega, \mathcal{A})$ such that*

(1) $\forall A, B \in \mathcal{A}$. $A \succsim^* B$ iff $\mu(A) \geq \mu(B)$, and

(2) $\forall A \in \mathcal{A}$ and $\rho \in [0, 1]$, there exists $B \in \mathcal{A}$, such that $\mu(B) = \rho\mu(A)$.

Proving this result requires some preparation first, which will come in the form of the following lemma.

Lemma 2. *Let $A, B, C, D \in \mathcal{A}$, and $\succsim^*$ is a divisible qualitative probability on $\mathcal{A}$. Then,*

1. *$B \subset C \implies \Omega \succsim^* C \succsim^* B \succsim^* \emptyset$.*
2. *$(\sim^*)$ $[A \sim^* B, A \cap C = \emptyset] \implies A \cup C \succsim^* B \cup C$.*
3. *$(\succ^*)$ $[A \succ^* B, A \cap C = \emptyset] \implies A \cup C \succ^* B \cup C$.*
4. *$(\sim^*)$ $[A \sim^* B, C \sim^* D, A \cap C = \emptyset] \implies A \cup C \succsim^* B \cup D$.*
5. *$(\succ^*)$ $[A \succsim^* B, C \succ^* D, A \cap C = \emptyset] \implies A \cup C \succ^* B \cup D$.*
6. *$[A \sim^* B, C \sim^* D, A \cap C = B \cap D = \emptyset] \implies A \cup C \sim^* B \cup D$.*
7. *$A \succ^* \emptyset \implies$ A can be partitioned into two events B and C for which $B \succ^* \emptyset$ and $C \succ^* \emptyset$.*
8. *$[A, B,$ and C are pairwise disjoint , $A \succsim^* B$, $B \cup C \succ^* A] \implies$ there exists $D \subset C$ such that $D \succ^* \emptyset$ and $B \cup (C/D) \succ^* A \cup D$.*
9. *$[A \succ^* \emptyset, B \succ^* \emptyset, A \cap B = \emptyset] \implies B$ can be partitioned into C and D such that $A \cup C \succsim^* D \succsim^* C$.*
10. *$A \succ^* \emptyset \implies A$ can be partitioned into B and C such that $B \sim^* C$.*
11. *$A \succ^* \emptyset \implies$ for any positive integer n there is a 2^n part partition of A such that $\sim^*$ holds between each two events n the partition.*

Proof. □

Now we get back to proving our result. Note that in view of part 11 of the lemma 2 above we shall call a partition $\{A_1, A_2, \ldots, A_n\}$ of A a u.p. (uniform partition) if $A \succ^* \emptyset$ and $A_1 \sim^* A_2 \sim^* \ldots \sim^* A_n$.

Proof of proposition 1. It is a long proof. So we shall divide into various steps.

Step 1. Suppose $\{A_1, A_2, \ldots, A_n\}$ is a n-part u.p. of Ω. Let $A \in \mathcal{A}$. Define

$$k(n, A) = max\{k | A \succsim^* \bigcup_{j=1}^{k} A_j\}.$$[2]

Then we show that the function $k(n, .)$ is well defined, that is $k(n, .)$ does not depend on the particular n-partition chosen. Let $\{A_1, A_2, \ldots, A_n\}$ is a u.p. of Ω such that $k(n, A) = k$. Let $\{B_1, B_2, \ldots, B_n\}$ be another u.p. of Ω. Then we claim that $B_i \sim^* A_j$ for all i, j. Suppose $B_{i'} \succ^* A_{i'}$ for some i', j'. Then, by transitivity, $B_i \succ^* A_j$ for all i, j. Since partitions are disjoint by definition, therefore by repeated use of lemma 2(5.) we get $\Omega \sim^* \bigcup_{j=1}^{m} B_j \succ^* \bigcup_{j=1}^{n} A_j \sim^* \Omega$, which gives us a contradiction. Thus, indeed we have $B_i \sim^* A_j$ for all i, j which gives us that $k(n, .)$ is well defined.

Step 2. Recall that for each integer n there exists a n-part u.p. of Ω. From here on we shall talk about 2^n-part partitions pf Ω. Let the function k be defined as above. Define a function $\mu : \mathcal{A} \to \mathbb{R}$ by

$$\mu(A) = \lim_{n \to \infty} \frac{k(2^n, A)}{2^n} \quad \forall A \in \mathcal{A}.$$

Note that for a fixed A, $\frac{k(2^n, A)}{2^n}$ is increasing in n. We claim that μ is a probability measure on $(\Omega, \mathcal{A})$ and $\mu(A) \geq \mu(B) \implies A \succsim^* B$. First we show that μ is a probability measure on $(\Omega, \mathcal{A})$.

(i) For a u.p. $\{A_1, A_2, \ldots, A_{2^n}\}$ of Ω, $\bigcup_{j=1}^{2^n} A_j \sim^* \Omega$, which implies $\Omega \succsim^* \bigcup_{j=1}^{2^n} A_j$. Thus, $\frac{k(2^n, \Omega)}{2^n} = 1$ for all n. Hence, $\mu(\Omega) = 1$.

(ii) By definition, $k(2^n, A) \geq 0$ for all n. Thus $\mu(A) \geq 0$.

(iii) Let A and B in $\mathcal{A}$ be such that $A \cap B = \emptyset$. We want to show that $\mu(A \cup B) = \mu(A) + \mu(B)$. First we shall develop a definition similar to $k(n, .)$. Define

$$l(n, A) = min\{l | \bigcup_{j=1}^{l} A_j \succ^* A\}$$

Note that as for k, we can symmetrically argue that $l(n, .)$ is well defined and $\frac{l(2^n, A)}{2^n}$ is decreasing in n.

We claim that $\lim_n \frac{k(2^n, A)}{2^n} = \lim_n \frac{l(2^n, A)}{2^n}$. It is clear by definitions that $\frac{k(2^n, A)}{2^n} \leq \frac{l(2^n, A)}{2^n}$. Thus, $\lim_n \frac{k(2^n, A)}{2^n} \leq \lim_n \frac{l(2^n, A)}{2^n}$. Suppose $\lim_n \frac{k(2^n, A)}{2^n} < \lim_n \frac{l(2^n, A)}{2^n}$. then, since $\{\frac{r}{2^n} | r = 0, 1, 2, \ldots, 2^n; n = 0, 1, 2, 3, \ldots\}$ is dense in $[0, 1]$, therefore for sufficiently large n, there exists r such that $\frac{k(2^n, A)}{2^n} < \frac{r}{2^n} < \frac{l(2^n, A)}{2^n}$. Since $\succsim^*$ is complete, therefore $A \succsim^* \bigcup_{j=1}^{r} A_j$ or $\bigcup_{j=1}^{r} A_j \succsim^* A$ which implies $\frac{k(2^n, A)}{2^n} \geq \frac{r}{2^n}$ or $\frac{l(2^n, A)}{2^n} \leq \frac{r}{2^n}$. Contradiction.

Now, we show that $\mu(A \cup B) \geq \mu(A) + \mu(B)$. For each n, let $k(2^n, A) = k_A^n$ and $k(2^n, B) = k_B^n$.

[2] Note that we could have equivalently said $k(n, A)$ is the largest number k such aah $A \succsim^* B$ where B is the union of k elements of the n-part u.p. But, by fact that all members of partition are indifferent and lemma 2 we can use take the first k scripted members of the u.p.

Then, by definition, $A \succsim^* \bigcup_{j=1}^{k_A^n} A_j$ and $B \succsim^* \bigcup_{j=1}^{k_B^n} A_j$. Now this is a delicate argument. Note that $A \cap B = \emptyset$ implies that $k_A^n + k_B^n \leq 2^n$. We can choose $I_A, I_B \subset \{1, 2, \ldots, 2^n\}$ such that $|I_A| = k_A^n, |I_B| = k_B^n$ and $I_A \cap I_B = \emptyset$. Thus, $A \supset \bigcup_{j \in I_A} A_j$ and $B \supset \bigcup_{j \in I_B} A_j$.[3]. Thus, we have $A \cup B \supset \bigcup_{j \in I_A \cup I_B} A_j$. By definition,

$$\begin{aligned}
k(2^n, A \cup B) &\geq |I_A \cup I_B| \\
&= |I_A| + |I_B| \\
&= k(2^n, A) + k(2^n, B) \\
\therefore \lim_n \frac{k(2^n, A \cup B)}{2^n} &\geq \lim_n \frac{k(2^n, A)}{2^n} + \lim_n \frac{k(2^n, B)}{2^n} \\
\text{i.e. } \mu(A \cup B) &\geq \mu(A) + \mu(B)
\end{aligned}$$

We can similarly show that $\mu(A) + \mu(B) \geq \mu(A \cup B)$ by playing around with $l(2^n, .)$ instead and then using the fact that $\mu(A) = \lim_n \frac{k(2^n, A)}{2^n} = \lim_n \frac{l(2^n, A)}{2^n}$.

Step 3. Now that we have a nice probability measure, we want to show that $A \succsim^* B$ iff $\mu(A) \geq \mu(B)$.

(i) $A \succsim^* B \implies \mu(A) \geq \mu(B)$. Suppose $A \succsim^* B$. Then for any k such that $B \succsim^* \bigcup_{j=1}^{k} A_j$, we have $A \succsim^* \bigcup_{j=1}^{k} A_j$. This implies, $k(2^n, A) \geq k(2^n, B)$ for all n. Thus, $\mu(A) \geq \mu(B)$.

(ii) $A \succ^* \emptyset \implies \mu(A) > 0$. Suppose $A \succ^* \emptyset$. Since $\succsim^*$ is divisible, therefore, there exists a partition $\{C_1, C_2, \ldots, C_m\}$ of Ω such that $A \succ^* C_i$ for all $i = 1, 2, \ldots, m$. This implies by *(i)* above that $\mu(A) \geq \mu(C_i)$. but then by finite additivity, we must have $\mu(A) \geq \mu(\Omega) = 1$. Thus, $\mu(A) > 0$.

(iii) $A \succ^* B \implies \mu(A) > \mu(B)$. Suppose $A \succ^* B$. We claim that there exists $C \in \mathcal{A}$ such that $C \succ^* \emptyset, B \cap C = \emptyset$, and $A \succ^* B \cup C$. Define $D = \Omega \backslash B$. Then, by part 1 of lemma 2 we know that $D \neq \emptyset$. Thus, $D \succsim^* \emptyset$. We claim that $D \succ^* \emptyset$. Suppose $D \sim^* \emptyset$. Then, by lemma 2(5.), we get $A \succ^* B \cup D = \Omega$, a contradiction. Thus $D \succ^* \emptyset$. Then, by lemma 2(**10.**), we know that D can be partitioned into C_1 and C_2 such that $C_1 \sim^* C_2 \succ^* \emptyset$. If $A \succ^* B \cup C_i$, we are done. If not, then partition one of the C_i's again and so on. Eventually we will get a set C such that $C \subset D$ and $A \succ^* B \cup C$. If we don't then it violates the fact that $\succsim^*$ is divisible.

Clearly *(i)* and *(ii)* imply $A \succsim^* B$ iff $\mu(A) \geq \mu(B)$.

Step 4. Let $A \in \mathcal{A}$ and $\rho \in [0, 1]$. We want to show that there exists $B \in \mathcal{A}$ such that $\mu(B) = \rho\mu(A)$. If $\mu(A) = 0$, the result is obvious, just choose $B = \emptyset$. So assume $\mu(A) > 0$ Now consider a sequence $\{A_1^1, A_2^1\}, \{A_1^2, \ldots, A_4^2\}, \ldots, \{A_1^n, \ldots, A_{2^n}^n\}, \ldots$ of 2^n-part u.p. of A for which

[3] It'll be nice to stare at lemma 2 and footnote 2 at this point.

$\{A^n_{2i-1}, A^n_{2i}\}$ is a 2-part u.p. of A^n_i.

$$A = A^1_1 \cup A^1_2$$
$$A^1_1 = A^2_1 \cup A^2_2, A^1_2 = A^2_3 \cup A^2_4$$
$$A^2_1 = A^3_1 \cup A^3_2, A^2_2 = A^3_3 \cup A^3_4, A^2_3 = A^3_5 \cup A^3_6, A^2_4 = A^3_7 \cup A^3_8$$
$$\vdots$$

Note that nth member of the sequence forms a 2^n-part partition of A. Now, for a given n let $m = \sup\{i|\mu(\cup^i_{j=1} A^n_j) < \rho\mu(A)\}$ so that

$$\mu(\cup^m_{j=1} A^n_j) + 2^{-n}\mu(A) \geq \rho\mu(A)$$

and let $k = \inf\{i|\mu(\cup^{2^n}_{j=i} A^n_j) < (1-\rho)\mu(A)\}$ so that

$$\mu(\cup^{2^n}_{j=k} A^n_j) + 2^{-n}\mu(A) \geq (1-\rho)\mu(A)$$

The two inequalities above are basically consequences of the definition of definition of supremum and infimum.

Next, let $C_n = \cup^m_{j=1} A^n_j$ and $D_n = \cup^{2^n}_{j=k} A^n_j$. Then by construction, $C_1 \subset C_2 \subset \ldots, D_1 \subset D_2 \subset \ldots$, and $C_i \cap D_j = \emptyset$ for all i, j. Also, $\mu(C_n) \geq \rho\mu(A) - 2^{-n}\mu(A)$ and $\mu(D) \geq (1-\rho)\mu(A) - 2^{-n}\mu(A)$. Taking limits on both sides of the inequalities, we get $\rho\mu(A) \leq \mu(\cup_n C_n)$ and $(1-\rho)\mu(A) \leq \mu(\cup_n D_n)$. Thus,

$$\mu(A) = \rho\mu(A) + (1-\rho)\mu(A) \leq \mu(\cup_n C_n) + \mu(\cup_n D_n)$$

Since $(\cup_n C_n) \cap (\cup_n D_n) = \emptyset$, thus by finite additivity

$$\begin{aligned} \mu(\cup_n C_n) + \mu(\cup_n D_n) &= \mu((\cup_n C_n) + (\cup_n D_n)) \\ &\leq \mu(A) \;\; (\because C_n, D_n \subset A \;\forall n) \end{aligned}$$

□

2 Constructing a lottery space on X

Document a simple result.

Lemma 3. *For all $A \in \mathcal{A}$, $\mu(A) = 0$ iff A is null.*

Proof. Follows from remark 1 and the fact that μ represents $\succsim^*$. □

Next we define our probability measure on the space of prizes X, which is induced by the measure μ we constructed above.

Definition 4. *For $f \in F$, let p_f on X be defined by $p_f = \mu \circ f^{-1}$.*

Let $F_0 = \{f \in F || f(\Omega)| < \infty\} \subset F$. That is, we define f to be the set of all simple acts on Ω. Also, as always let $\mathcal{L}$ be the set of probability measures on X.[4] Let $\mathcal{L}_0$ be the set of simple probability measures (lotteries) on X.

Proposition 2. *For all $f \in F_0$, $p_f \in \mathcal{L}_0$.*

Proof. □

3 Constructing a preference relation on $\mathcal{L}_0$

In order to construct a preference on $\mathcal{L}_0$, we want to show that L is "good" mapping. Next, we show that L is onto and is one-one only up to indifference. That is, $L(f) = L(g) \implies f \sim g$.

Proposition 3. *$L : F_0 \to \mathcal{L}_0$ is onto, and $(Lf) = L(g) \implies f \sim g$.*

The proof for onto is relatively straightforward, but for one-one is quite detailed. We shall use two lemmas in proving the latter result.

Lemma 4. *Let $A \in \mathcal{A}$ and $\{A_1, A_2, \ldots, A_n\}$ be a partition of A. Then, $fA_ih \succsim gA_ih$ for all $i = 1, \ldots, n \implies fAh \succsim gAh$.*

Proof. □

Lemma 5. *Let $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$. Then, $xA(yBh) \sim yA(xBh)$.*

Proof. □

Proof of proposition 3. □

Next we do the important step of defining a preference relation on $\mathcal{L}_0$. It will be induced by preference relation $\succsim$ on F.

Proposition 4. *For any $p, q \in \mathcal{L}$, let $p \succsim_l q$ iff $f \succsim g$ for all $f, g \in F_0$ such that $p = p_f$ and $q = p_g$. Then $\succsim_l$ is a well defined preference relation on $\mathcal{L}_0$.*

Proof. Let $p, q \in \mathcal{L}_0$. Since L is onto, therefore, $\exists f$ and g in F_0 such that $p = p_f$ and $q = p_g$. Also, by theorem 3 it is clear that if there is any other $f' \in F_0$ such that $p = p_{f'}$, then $f \sim f'$ (similarly for g). Thus, the binary relation is indeed well defined. In fact to be clear on that, with a slight abuse of notation, we can write $p \succsim_l q$ iff $L^{-1}(p) \succsim L^{-1}(q)$.

Next we show that $\succsim_l$ is a preference relation. Let $p = p_f$ and $q = p_g \in \mathcal{L}_0$. Since $\succsim$ is complete, we have $f \succsim g$ or $g \succsim f$. Thus, by definition, $p \succsim_l q$ or $q \succsim_l p$. Next, let $p = p_f$, $q = p_g$ and $r = p_h \in \mathcal{L}_0$ such that $p \succsim_l q$ and $q \succsim_l r$. Then $f \succsim g$ and $g \succsim h$. Since $\succsim$ is transitive, $f \succsim h$. Thus, $p \succsim_l r$. □

[4] more precisely on the algebra of the set of all subsets on X.

4 Invoking the Mixture Space Theorem

Once we have constructed a binary relation on $\mathcal{L}_0$, we are ready to invoke the Mixture Space Theorem. It is easy to record that $(\mathcal{L}_0, +, .)$ is a mixture space. The following lemma establishes that the three axioms for the mixture space theorem hold.

Lemma 6. *$\succsim_l$ on $\mathcal{L}_0$ satisfies the mixture space axioms:*

1. *$\succsim_l$ is a preference relation.*
2. *For all $p, q, r \in \mathcal{L}_0$, $p \succ_l q$ and $a \in (0,1) \implies ap + (1-a)r \succ_l aq + (1-a)r$.*
3. *For all $p, q, r \in \mathcal{L}_0$, $p \succ_l q \succ_l r \implies \exists\ a, b \in (0,1)$ such that $ap + (1-a)r \succ_l q \succ_l bp + (1-b)r$.*

Now for the final argument. Since $(\mathcal{L}_0, +, .)$ is a mixture space that satisfies the three axioms, we have that $\succsim_l$ admits a linear representation, say U, on $\mathcal{L}_0$. Thus, there exists $u : X \to \mathbb{R}$ such that $U(p) = \sum_{x \in X} u(x)p(x)\ \forall\ p \in \mathcal{L}_0$. Next, define $W = U \circ L$, so that

$$W(f) = \sum_{x \in X} u(x)L(f(x)) = \sum_{x \in X} u(x)p_f(x) = \sum_{x \in X} u(x)\mu(f^{-1}(x))$$

Thus, W is "linear". Moreover,

$$\begin{aligned} f \succsim g &\Leftrightarrow L(f) \succsim_l L(g) && \text{(by definition of } \succsim_l) \\ &\Leftrightarrow U \circ L(f) \geq U \circ L(g) && \text{(since } U \text{ represents } \succsim_l) \\ &\Leftrightarrow W(f) \geq W(g) && \text{(by definition of } W) \end{aligned}$$

Hence, W represents $\succsim$.