Anscombe and Aumann Theorem

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1 Preliminaries

So far we have treated probabilities over prizes as being given. In other words we have been only dealing with objective probabilities. Savage argues that such a description offers limited insight into the choice theoretic behaviors of individuals since the probabilities are assumed to be set by nature or some exogenous elements. Let us consider a simple example to motivate Savage's idea of subjective probabilities.

Example 1. *Suppose there is a tri-nation cricket tournament between India, Pakistan and Australia. You are asked the following question: Which team is most probable to win the tournament> You answer India. Then you are asked the question: Which scenario do you think is more likely -[Pakistan or Australia will win] or [India will win]? You answer the former is more likely! What can be inferred about your subjective probabilities from these choices? If certain intuitive assumptions are satisfied, it is easy to see that the choices expressed are consistent with-* $\frac{1}{2} > p(India) > \frac{1}{3}$*, where* $p(India)$ *is the probability that India will win the tournament.*

In the savage style models, uncertainty its viewed as being subjective in the sense that there are no objective (externally) imposed probabilities. Probabilities will enter the story, being part of the eventual representation, but they will be supplied by the decision maker (DM) on the basis of his subjective preferences. The basics of the Savage formulation are:

(i) a set of prizes or consequences, denoted X, and

(ii) a set of *states of the world or of nature* denoted by Ω with typical element s. Each $s \in \Omega$ is a compilation of all characteristics/factors about which the DM is uncertain and which are relevant to the consequences that will ensue form his choice. The set Ω is to be an exhaustive list of mutually exclusive states.

The Savage formulation is a remarkable achievement both on philosophical and modeling grounds. We will discuss more of it later. But, is it necessary to go as far as Savage and have all uncertainty as subjective? Can't we agree (if only as a "thought experiment") that there are objective randomizing devices such as fair coins, perfect dice, balanced roulette wheels, urns filled with colored balls, etc.?

If we can, then a lot of difficulty that Savage encounters can be eliminated using a "middle of the road" formulation and development due to Anscombe and Aumann.

Anscombe-Aumann setup. The setup starts the same way as Savage with X and Ω. Let X be an arbitrary set and Ω be finite. AS before $\mathcal{L}(X)$ is the set of simple probability distributions on X. Since the context is obvious, we use just $\mathcal{L}$ to denote this set. The choice space is given by $H = \mathcal{L}^{\Omega}$. The representation we seek is:

There exists functions $\pi : \Omega \to [0, 1]$ with $\sum_{s\in\Omega} \pi(s) = 1$ and $u : X \to \mathbb{R}$ such that

$$h \succsim h' \text{ iff } \sum_{s\in\Omega} \pi(s)\Big[\sum_{x\in X} h(s)(x)u(x)\Big] \geq \sum_{s\in\Omega} \pi(s)\Big[\sum_{x\in X} h'(s)(x)u(x)\Big] \qquad (\wp)$$

Note that for each $s \in \Omega$, $h(s)$ is a distribution on X. Also for notational convenience let $\Omega = \{1, 2 \ldots, n\}$. Then H is the set of all functions $h : \Omega \to \mathcal{L}$. We shall write $h(s)$ and h_s interchangeably so that every $h \in H$ has the form

$$h = \{h_1, h_2, \ldots, h_n\}$$

where $h_s \in \mathcal{L}$ for each $s = 1, 2, \ldots, n$.

each $h \in H$ is a compound lottery- the horse race is run and if outcome is s, the randomizing devices are used to construct simple probability distribution h_s.

2 State Dependent Representation

For h and g in H and $a \in [0, 1]$, define $ah + (1 - a)g$ by

$$(ah + (1 - a)g)(s) = ah(s) + (1 - a)g(s) \ \forall s \in \Omega$$

That is two compound lotteries are "mixed" by mixing the objective lotteries that comprise them. Note that $(H, +, .)$ forms a mixture space. Revisiting the axioms:

Axiom A1: $\succsim$ on H is a preference relation.

Axiom A2: For all $h, h', g \in H,\ h \succ h'$ and $a \in (0, 1) \implies ah + (1 - a)g \succ ah' + (1 - a)g$.

Axiom A3: For all $h, h', h'' \in H,\ h \succ h' \succ h''' \implies \exists a, b \in (0, 1)$ such that $ah + (1 - a)h'' \succ h' \succ bh + (1 - b)h''$.

We can now state the following theorem:

Theorem 1. *Let $\succsim$ be a binary relation on H. Then $\succsim$ satisfies Axioms A1-A3 iff there exists function $u_s : X \to \mathbb{R}$ for each $s = 1, 2 \ldots, n$ such that*

$$h \succsim g \text{ iff } \sum_{s=1}^{n} \sum_{x\in X} h_s(x)u_s(x) \geq \sum_{s=1}^{n} \sum_{x\in X} g_s(x)u_s(x) \qquad (\ltimes)$$

Moreover, if $u_1', u_2', \ldots, u_n'$ is another collection of function satisfying the above condition, then there exist constants $a > 0$ and $b_s > 0$ such that $u_s' = au_s + b_s$ for each s.

Proof. That such a representation implies the three axioms hold is simple and can be done as before in vNM.

Assume Axioms A1-A3 hold. since H is a mixture space, we know by the Mixture Space Theorem that there exists a function $U : H \to \mathbb{R}$ such that

$$h \gtrsim g \text{ iff } U(h) \geq U(g). \text{ and,}$$
$$U(ah + (1-a)g) = aU(h) + (1-a)U(g) \; \forall a \in [0,1]$$

Moreover, this U is unique unto positive affine transformations. We will now show that it has the form

$$U(h) = \sum_{s=1}^{n} \sum_{x \in X} h_s(x) u_s(x).$$

for some functions $u_1, u_2, \ldots, u_n$. To do this, fix some $h^* \in H$. For any $h \in H$ let $h^1 = (h_1, h_2^*, \ldots, h_n^*)$, $h^2 = (h_1^*, h_2, \ldots, h_n^*)$, etc. That is h^s is h^* but with h_s replacing h_s^*. Observe that

$$\frac{1}{n}h + \frac{n-1}{n}h^* = \sum_{s=1}^{n} \frac{1}{n} h^s \tag{$\ominus$}$$

Thus by the linearity of U and standard induction arguments, we get

$$\frac{1}{n}U(h) + \frac{n-1}{n}U(h^*) = \sum_{s=1}^{n} \frac{1}{n} U(h^s). \tag{$\diamond$}$$

For $s = 1, 2, \ldots, n$define $U_s : \mathcal{L} \to \mathbb{R}$ by

$$U_s(p) = U(h_1^*, \ldots, h_{s-1}^*, p h_{s+1}^*, \ldots, h_n^*) - \frac{n-1}{n}U(h^*) \tag{$\dagger$}$$

Thus for $h \in H$, this definition gives

$$U_s(h_s) = U(h^s) - \frac{n-1}{n}U(h^*)$$

Summing this last equation over s and dividing by n yields,

$$\frac{1}{n}\sum_{s=1}^{n} U_s(h_s) = \frac{1}{n}\sum_{s=1}^{n} U(h^s) - \frac{n-1}{n}U(h^*)$$

Comparing this with (⋄), we have

$$U(h) = \sum_{s=1}^{n} U_s(h_s)$$

Now (†) and linearity of U yield

$$U_s(ap + (1-a)q) = aU_s(p) + (1-a)U_s(q).$$

Finally, for each $x \in X$, define $u_s(x) = U_s(\delta_x)$- then the usual induction argument, using the fact that the support of $\mathcal{L}$ is finite, like in proposition ??, shows that

$$U_s(p) = \sum_{x \in X} p(x)U_s(\delta_x) = \sum_{x \in X} p(x)u_s(x).$$

Thus,

$$\begin{aligned} U(h) &= \sum_{s=1}^{n} U_s(h_s) \text{ (shown above)} \\ &= \sum_{s=1}^{n} [\sum_{x \in X} h_s(x)u_s(x)] \end{aligned}$$

This establishes the desired representation. □

Remark 1. *Note that basic representation follows immediately from the Mixture Space Theorem. The challenge really was to get a characterization of the kind we managed in Proposition ??. We could call this characterization the state dependent linear representation!*

3 State Independent Representation

Go back to section 3.1 and stare at identity ℘. Now proceed to theorem 1 and look at expression ⋉. We did not exactly prove what we had set out to do. Where is the disconnect? It is the u with and without the subscripts! We want there to be a single function $u : X \to \mathbb{R}$ and a probability distribution μ on the states of the world $\Omega = \{1, 2, \ldots, n\}$ such that

$$h \succsim g \text{ iff } \sum_{s=1}^{n} \mu(s)\Big[\sum_{x \in X} h_s(x)u(x)\Big] \geq \sum_{s=1}^{n} \mu(s)\Big[\sum_{x \in X} g_s(x)u(x)\Big] \qquad (\wp')$$

Since expression ⋉ is necessary and sufficient for the three mixture space axioms, we need to go looking for more axioms. First rule out the trivial case where $h \sim g$ for all $h, g \in H$.

Axiom A4: There exists h and g in H such that $h \succ g$.

Next consider the following definition.

Definition 1. *State s is said to be Null if $h \sim g$ for ll pairs h and g such that $h_{s'} = g_{s'}$ for all $s' \neq s$.*

That is, if we can't find compound lotteries that differ only in the sth component and that are not indifferent to each other, then the sth state of nature can be ignored and is called null. Note the following corollary as a result of this definition.

Corollary 1. *Let $\succsim$ satisfy Axioms A1-A3 and let $\{u_s, s = 1, 2, \ldots, n\}$ represent $\succsim$ in the sense of expression $\ltimes$. Then state s is null iff u_s is constant on X.*

The key to getting expression $\wp'$ from expression $\ltimes$ is the following axiom.

Axiom A5: If $h \in H, p, q \in \mathcal{L}$ are such that

$$(h_1, \ldots, h_{s-1}, p, h_{s+1}, \ldots, h_n) \succsim (h_1, \ldots, h_{s-1}, q, h_{s+1}, \ldots, h_n)$$

for some s, then for all non-null s',

$$(h_1, \ldots, h_{s'-1}, p, h_{s'+1}, \ldots, h_n) \succsim (h_1, \ldots, h_{s'-1}, q, h_{s'+1}, \ldots, h_n).$$ [1]

That is, if p is better q in state s, then p is better than q in all non-null states s'. This is a very strong axiom and will fail to hold most applications.

Example 2. *Suppose that states are possible weather types instead of results of horse races and prizes are bundles of picnic equipment. To be precise suppose $\Omega = \{shine, rain\}$, and $p = \delta_x, q = \delta_{x'}$, where x and x' are identical bundles of equipment except that x has an umbrella and x' does not. Presumably DM strictly prefers to have x to x' in state 2 (rain), but is at least indifferent in state 1. Of course in this example we wouldn't (or rather shouldn't) expect a state independent expected utility representation-tastes are quite clearly state dependent.*

Theorem 2. *Axioms A1-A5 are necessary and sufficient for there to exist a non constant utility function $u : x \to \mathbb{R}$ and a probability distribution μ on Ω such that*

$$h \succsim g \ \text{ iff } \ \sum_{s=1}^{n} \mu(s) \Big[\sum_{x \in X} h_s(x) u(x) \Big] \geq \sum_{s=1}^{n} \mu(s) \Big[\sum_{x \in X} g_s(x) u(x) \Big]$$

Moreover, the probability distribution is unique, and u is unique up to a positive affine transformation in this representation.

Proof. Suppose the axioms hold. Then A1-A3 imply a representation in the form of ($\ltimes$). Moreover, by A4 there is atleast one non-null state-let s^0 be one such. Take $p, q \in \mathcal{L}$ and let s be a non-null

[1]Prof. Gul introduces this cool notation. For each $s \in \Omega$ and $p \in \mathcal{L}$, let $h^{sp} = (h_1, \ldots, h_{s-1}, p, h_{s+1}, \ldots, h_n)$. Then the axiom becomes, if $h^{sp} \succsim h^{sq}$ for some s, then $h^{s'p} \succsim h^{s'q}$ for all non-null s'.

state. Then by the state dependent representation, for arbitrary h,

$$\begin{aligned}\sum_{x\in X} u_s(x)p(x) \geq \sum_{x\in X} u_s(x)q(x) &\text{ iff } h^{sp} \gtrsim h^{sq} \\ &\text{ iff } h^{s^0 p} \gtrsim h^{s^0 q} \text{ (by A5)} \\ \text{iff } \sum_{x\in X} u_{s^0}(x)p(x) \geq \sum_{x\in X} u_{s^0}(x)q(x) &\end{aligned}$$

By the uniqueness result for vNM expected utility for simple lotteries this says that there exists constants $a_s > 0$ and b_s such that

$$a_s u_{s^0}(.) + b_s = u_s(.).$$

For the null states such constants exist as well- but with $a_s = 0$, since s is null if and only if u_s is constant. So if we define $u(.) = u_{s^0}(.)$, $(\ltimes)$ becomes

$$h \gtrsim g \text{ iff } \sum_{s=1}^{n} \sum_{x\in X} h_s(x)(a_s u(x) + b_s) \geq \sum_{s=1}^{n} \sum_{x\in X} g_s(x)(a_s u(x) + b_s).$$

which simplifies to

$$h \gtrsim g \text{ iff } \sum_{s=1}^{n} [b_s + a_s(\sum_{x\in X} h_s(x)u(x))] \geq \sum_{s=1}^{n} [b_s + a_s(\sum_{x\in X} g_s(x)u(x))]$$

Cancel the identical b_s on both sides.[2] Divide both sides by the strictly positive term $\sum_s a_s$ and then define $\mu(s) = \frac{a_s}{\sum_s a_s}$. We have $(\wp')$ as desired.

(Still have to show converse and uniqueness.) □

[2]Who needs BS anyway?