Expected utility and Mixture space theorem

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1 Preliminaries and linear representation

Let X be a finite set of prizes. Let $\mathcal{L}(X)$ be the set of lotteries on X, i.e.,

$$\mathcal{L}(X) = \{p : X \to [0, 1] | \sum_{x \in X} p(x) = 1\}$$

To simplify notation[1] we will use $\succsim$ for a preference relation and $\succ$ for a K-preference relation and also use there for the corresponding derived binary relationships (e.g. $\succ_{\succsim} \equiv \succ$ etc.). The context should be clear in each case.

For any $x \in X$, δ_x denotes the lottery that chooses x with certainty. That is ,

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Let any $p, q \in \mathcal{L}(X)$ and $a \in [0, 1]$. Then $ap + (1 - a)q \in \mathcal{L}(X)$ and is often referred to as a compound lottery. Dixit writes

> A general presumption of standard microeconomic theory is that people care only about what they finally get to consume, and not by the path or process by which they arrived at this consumption vector. This is sometimes called the compound lottery axiom or the reduction of compound lotteries. This axiom will be violated if the decision-maker likes or dislikes the process by which uncertainty is resolved; for example he may enjoy the suspense that builds up as a lottery yields a prize that is another lottery, or he may find it worrying.

In this chapter, we will assume that the compound lottery axiom.

We say that $U : \mathcal{L}(X) \to \mathbb{R}$ is linear if

$$U(ap + (1 - a)q) = aU(p) + (1 - a)U(q) \;\; \forall p, q \in \mathcal{L}(X) \text{ and } a \in [0, 1]$$

We say that U is a linear representation of $\succsim$ if U is linear and represents $\succsim$.

[1] Prof. Gul says "The gloves are now off!"

Proposition 1. *If U is a linear function on $\mathcal{L}(X)$, then there exists a real-valued function u on X such that $U(p) = \sum_{x\in X} p(x)u(x)$ for all $p \in \mathcal{L}(X)$.*

Proof. For any $x \in X$ define $u(x) = U(\delta_x)$. Further, for any $p \in \mathcal{L}(X)$, define $S_p = \{x \in X | p(x) > 0\}$. S_p is basically the support of p. We prove the result by induction on $|S_p|$.

If $|S_p| = 1$, then the claim is trivial. We have $u(x) = U(\delta_x)$ where $\{x\} = S_p$. Suppose we have shown the result for $|S_p| = k$. We claim that it holds for $|S_p| = k + 1$.

Fix any $x \in X$. Note that, any $p \in \mathcal{L}(X)$ can be written as,

$$p = p(x)\delta_x + (1 - p(x))q$$

where

$$q(z) = \begin{cases} 0 & \text{if } z = x \\ \frac{p(z)}{1-p(x)} & \text{if } z \neq x. \end{cases}$$

Clearly, $q \in \mathcal{L}(X)$ and $|S_q| = k$. Therefore, by the induction hypothesis, $U(q) = \sum_{z\in x} q(z)U(\delta_z)$. Thus,

$$U(p) = p(x)U(\delta_x) + (1 - p(x))U(q) = \sum_{z\in X} p(z)U(\delta_z)$$

and we are done. □

If X was an arbitrary srt, the above construction works if we consider $\mathcal{L}(X)$ to be the set of all simple lotteries on X.[2] Note that even if X is countably infinite and we consider non-simple lotteries, the above construction does not work.

2 von Neumann-Morgenstern Expected Utility

von Neumann-Morgenstern were one of the first to provide a full theory of expected utility in an axiomatic fashion. It is important to note that the vNM model views uncertainty as objective, in the sense that there is given a quantification of how likely the various outcomes are, given in the form of a probability distribution. The other view by Savage regards uncertainty as subjective, which we will discuss later.

Axiom 1: $\succsim$ is a preference relation.

Axiom 2: For all $p, q, r \in \mathcal{L}(X)$, $p \succ q$ and $a \in (0, 1) \implies ap + (1 - a)r \succ aq + (1 - a)r$.

Axiom 3: For all $p, q, r \in \mathcal{L}(X)$, $p \succ q \succ r \implies \exists\, a, b \in (0, 1)$ such that $ap + (1 - a)r \succ q \succ bp + (1 - b)r$.

[2] A simple lottery is a lottery with a finite support.

Axiom 2 is often called the independence or substitution axiom, and Axiom 3 is often called the Archimedean or continuity axiom.

Theorem 1. *Let X be a finite set and $\succsim$ be a binary relation on $\mathcal{L}(X)$. Then, $\succsim$ satisfies Axioms 1-3 iff it has a linear representation.*

Proof. We break the proof into the following lemmas.

Lemma 1. *If $\succsim$ satisfies Axioms 1-3, then:*

(i) $p \succ q$ *and* $0 \leq a < b \leq 1 \implies bp + (1-b)q \succ ap + (1-a)q$

(ii) $p \succsim q \succsim r$ *and* $p \succ r \implies \exists$ *a unique* $a^* \in [0,1]$ *such that* $q \sim a^* p + (1-a^*)r$

(iii) $p \sim q$ *and* $a \in [0,1] \implies ap + (1-a)r \sim aq + (1-a)r \ \forall r \in \mathcal{L}(X)$.

Proof of Lemma 1. (i) First consider the special case $a = 0$. Then, $p \succ q$ and $0 \leq b \leq 1$ with Axiom 2 imply $bp + (1-b)q \succ bq + (1-b)q = q = ap + (1-a)q$. Now let $r = bp + (1-b)q$ and suppose $a > 0$. Then, $(a.b) < 1$, and $r \succ q$ and Axiom 2 together imply that

$$\begin{aligned} r &= (1-\frac{a}{b})r + \frac{a}{b}r \\ &\succ (1-\frac{a}{b})q + \frac{a}{b}r \\ &= (1-\frac{a}{b})q + \frac{a}{b}(bp + (1-b)q) \\ &= ap + (1-a)q. \end{aligned}$$

(ii) Since $p \succ r$, part *(i)* ensures that if a^* exists, it is unique. If $p \sim q$, then $a^* = 1$ works. If $q \sim r$, then $a^* = 0$ works. So we need to only consider the case $p \succ q \succ r$. Define

$$a^* = sup\{a \in [0,1] | q \succsim ap + (1-a)r\}$$

Since $a = 0$ is in the set, we know we aren't taking supremum over an empty set.

By definition of a^*, if $1 \geq a \geq a^*$, then $ap + (1-a)r \succ q$. Moreover by *(i)*, if $0 \leq a \leq a^*$, then $q \succ ap + (1-a)r$. To see this, note that if $0 \leq a \leq a^*$, then there exists a' such that $0 \leq a \leq a' \leq a^*$ and $q \succsim a'p + (1-a')r$ by definition of a^*. And $a < a'$ implies that $q \succsim a'p + (1-a')r \succ ap + (1-a)r$.

There are three possibilities to consider.

Suppose $a^*p + (1-a^*)r \succ q \succ r$. Then by Axiom 3, there exists $b \in (0,1)$ such that $b(a^*p + (1-a^*)r) + (1-b)r \succ q$, that is, $ba^*p + (1-ba^*)r \succ q$. But $ba^* < a^*$, so we have $q \succ ba^*p + (1-ba^*)r$. Contradiction.

Suppose $p \succ q \succ a^*p + (1-s^*)r$. Then, by Axiom 3, there exists $b \in (0,1)$, such that $q \succ b(a^*p + (1-a^*)r) + (1-b)p$, that is, $q \succ (1-b(1-a^*))p + b(1-a^*)r$. Since, $(1-b(1-a^*)) > a^*$, we have from the above argument that $(1-b(1-a^*))p + b(1-a^*)r \succ q$. Contradiction.

This leaves us with the third possibility (which is what we want), namely $q \sim a^*p + (1-a^*)r$.

(iii) This result is trivial for the case where, for all $s \in \mathcal{L}(X)$, $p \sim q \sim s$. So suppose there is some $s \in \mathcal{L}(X)$ such that $s \succ p \sim q$. (The other case where there is some s such that $p \sim q \succ s$

can be similarly done.) Let any $r \in \mathcal{L}(X)$ and $a \in [0,1]$. We want to show that $ap + (1-a)r \sim aq + (1-a)r$. Suppose $ap + (1-a)r \succ aq + (1-a)r$. From Axiom 2 we can deduce that for all $b \in (0,1)$, $bs + (1-b)q \succ bq + (1-b)q = q \sim p$. Using Axiom 2 again, we get that $a(bs + (1-b)q) + (1-a)r \succ ap + (1-a)r$, for all $b \in (0,1)$. Since (by assumption) $ap + (1-a)r \succ aq + (1-a)r$, Axiom 3 implies that for each b there exists some $a^*(b)$ such that $ap + (1-a)r \succ a^*(b)(a(bs + (1-b)q) + (1-a)r) + (1-a^*(b))(aq + (1-a)r)$. Fix, say $b = 1/2$, and let $a^*(1/2)$ be written a^*; then we have that

$$ap + (1-a)r \succ [a^*a/2]s + [a^*a/2 + (1-a^*)a]q + [1-a]r$$

But the term on the right hand side is

$$a[(a^*/2)s + (1-a^*/2)q] + (1-a)r$$

and since $a^*/2$, this must be $\succ ap + (1-a)r$, a contradiction. The case where we start by assuming $aq + (1-a)r \succ ap + (1-a)r$ will similarly lead to a contradiction. Thus, we must have $ap + (1-a)r \sim aq + (1-a)r$. □

Lemma 2. *If $\succsim$ on $\mathcal{L}(X)$ satisfies Axioms 1-3, then there exists $\bar{z}$ and $\underline{z}$ in X such that $\delta_{\bar{z}} \succsim p \succsim \delta_{\underline{z}}$ for all $p \in \mathcal{L}(X)$.*

Proof. Since$X$ is finite, there exists $\bar{z}$ and $\underline{z}$ in X such that $\delta_{\bar{z}} \succsim \delta_x \succsim \delta_{\underline{z}}$ for all $x \in X$. Let any $p \in \mathcal{A}(X)$. We prove this by induction on the cardinality of S_p, where $S_p = \{x \in X | p(x) > 0\}$.

If $|S_p| = 1$, then $p = \delta_x$ for some $x \in X$, and we already know $\delta_{\bar{z}} \succsim \delta_x \succsim \delta_{\underline{z}}$. Suppose we have shown the result for $|S_p| = k$. We claim that it also holds for $|S_p| = k+1$.

Fix any $x \in X$. Note that p can be written as,

$$p = p(x)\delta_x + (1-p(x))q$$

where

$$q(z) = \begin{cases} 0 & \text{if } z = x \\ \frac{p(z)}{1-p(x)} & \text{if } z \neq x \end{cases}$$

Clearly, $q \in \mathcal{L}(X)$ and $|S_q| = k$. Therefore, by induction hypothesis, $\delta_{\bar{z}} \succsim q$. Now $\delta_{\bar{z}} \succsim \delta_x \implies \delta_{\bar{z}} \succ \delta_x$ or $\delta_{\bar{z}} \sim \delta_x$, and $\delta_{\bar{z}} \succsim q \implies \delta_{\bar{z}} \succ q$ or $\delta_{\bar{z}} \sim q$. Then using Axiom 2 and lemma 1*(iii)* we can get

$$\begin{aligned} \delta_{\bar{z}} &= p(x)\delta_{\bar{z}} + (1-p(x))q \\ &\succsim p(x)\delta_x + (1-p(x))q \\ &= p \end{aligned}$$

Thus $\delta_{\bar{z}} \succsim p$. Similarly we can show that $p \succsim \delta_{\underline{z}}$.

□

Now we shall use the above lemmas to prove the main theorem. Suppose $\succsim$ satisfies Axioms 1-3. Apply lemma 2 to produce $\delta_{\bar{z}}$ and $\delta_{\underline{z}}$. If $\delta_{\bar{z}} \sim \delta_{\underline{z}}$, then set $U(p) = \pi$ for all $p \in \mathcal{L}(X)$, and we are done. So for the rest of the proof we consider the case $\delta_{\bar{z}} \succ \delta_{\underline{z}}$. For $p \in \mathcal{L}(X)$, define,

$$U(p) = a \text{ where } a\delta_{\bar{z}} + (1-a)\delta_{\underline{z}} \sim p$$

By lemma 1*(ii)* such an a exists and is unique, so U is well defined. We claim that this U will do the job for us.

For all $p, q \in \mathcal{L}(X)$ and $a \in [0, 1]$, by applying lemma 1*(iii)* twice, we get

$$\begin{aligned} ap + (1-a)q &\sim a[U(p)\delta_{\bar{z}} + (1-U(p))\delta_{\underline{z}}] + (1-a)[U(q)\delta_{\bar{z}} + (1-U(q))\delta_{\underline{z}}] \\ &= [aU(p) + (1-a)U(q)]\delta_{\bar{z}} + [1 - aU(p) - (1-a)U(q)]\delta_{\underline{z}} \end{aligned}$$

Thus by definition of U,

$$U(ap + (1-a)q) = aU(p) + (1-a)U(q).$$

This shows that U is linear.

Finally we want to show that U represents $\succsim$. We claim that

$$U(p) \geq U(q) \text{ iff } U(p)\delta_{\bar{z}} + (1-U(p))\delta_{\underline{z}} \succsim U(q)\delta_{\bar{z}} + (1-U(q))\delta_{\underline{z}}$$

If $U(p) = U(q)$, then the two lotteries are equal. Suppose $U(p) > U(q)$. Then by lemma 1*(i)*, we have $U(p)\delta_{\bar{z}} + (1-U(p))\delta_{\underline{z}} \succ U(q)\delta_{\bar{z}} + (1-U(q))\delta_{\underline{z}}$.

Conversely, suppose $U(p)\delta_{\bar{z}} + (1-U(p))\delta_{\underline{z}} \succsim U(q)\delta_{\bar{z}} + (1-U(q))\delta_{\underline{z}}$. If we were to have $U(q) > U(p)$, then by lemma 1*(i)* would give us $U(q)\delta_{\bar{z}} + (1-U(q))\delta_{\underline{z}} \succ U(p)\delta_{\bar{z}} + (1-U(p))\delta_{\underline{z}}$, which is a contradiction. Thus, we must have $U(p) \geq U(q)$, which completes our claim.

Now suppose $\succsim$ has a linear representation. That is, there exists $U : X \to \mathbb{R}$ such that

$$\begin{aligned} U(ap + (1-a)q) = aU(p) + (1-a)U(q) \;\; \forall p, q \in \mathcal{L}(X) \;\text{ and } a \in [0, 1] \\ \text{and } p \succsim q \text{ iff } U(p) \geq U(q) \;\; \forall p, q \in \mathcal{L}(X) \end{aligned}$$

We want to show that Axioms 1-3 hold.

In the following discussion, let p, q, r be any elements in $\mathcal{L}(X)$.

Consider p and q. Clearly, $U(p) \geq U(q)$ or $U(q) \geq U(p)$, that is, $p \succsim q$ or $q \succsim p$. Next, let $p \succsim q$ and $p \succsim r$. Then, $U(p) \geq U(q) \geq U(r)$. Thus, $U(p) \geq U(r)$ which implies $p \succsim r$. Thus $\succsim$ is complete and transitive and hence a preference relation. Axiom 1 is satisfied.

Next suppose $p \succ q$ and let $a \in (0, 1)$. Then $U(p) > U(q)$. Now, $U(ap + (1-a)r) = aU(p) + (1-a)U(r)$ and $U(aq + (1-a)r) = aU(q) + (1-a)U(r)$. Also, $U(p) > U(q)$ implies

$aU(p) + (1-a)U(r) > aU(q) + (1-a)U(r)$, that is $U(ap + (1-a)r) > U(aq + (1-a)r)$. Hence, $ap + (1-a)r \succ aq + (1-a)r$ which satisfies Axiom 2.

Finally, suppose $p \succ q \succ r$. Then, $U(p) > U(q) > U(r)$. Note that $t \in [0,1]$,

$$tU(p) + (1-t)U(r) > U(q) \text{ iff } t > \frac{U(q) - U(r)}{U(p) - U(r)}, \text{ and}$$
$$tU(p) + (1-t)U(r) < U(q) \text{ iff } t < \frac{U(q) - U(r)}{U(p) - U(r)}$$

Therefore, define

$$a = \frac{U(p) - U(r) + \epsilon}{U(p) - U(r)} \text{ and } b = \frac{U(q) - U(r) - \epsilon}{U(q) - U(r)}$$

where ϵ is chosen small enough so that $a, b \in (0,1)$. This will do the job. Thus, Axiom 3 is also satisfied. □

Theorem 2. *If $U : \mathcal{L}(X) \to \mathbb{R}$ represents $\succsim$, then a function $V : \mathcal{L}(X) \to \mathbb{R}$ represents $\succsim$ iff there exists real numbers $c > 0$ and b such that*

$$V(p) = cU(p) + d \; \forall p \in \mathcal{L}(X)$$

Proof. □

3 Mixture Space Theorem

This section is based mostly on the classic paper by Herstein and Milnor [1953].

Definition 1. *Let Π be any set with two operations, viz. $\oplus$ and $\circ$, defined on it such that for eery $p, q \in \Pi$ and $a, b \in [0,1]$, we have $a \circ p \oplus (1-a) \circ q \in \Pi$, and the following properties are satisfied:*

(1) $1 \circ p \oplus (1-1) \circ q = p$.

(2) $a \circ p \oplus (1-a) \circ q = (1-a) \circ q \oplus a \circ p$, *and*

(3) $b \circ [a \circ p \oplus (1-a) \circ q] \oplus (1-b) \circ q = (ab) \circ p \oplus (1-ab) \circ q$.

Then, $(\Pi, \oplus, \circ)$ *is called a Mixture Space.*

Note that $\oplus$ corresponds to the addition operation of real numbers and $\circ$ corresponds to the multiplication operation of the real numbers. In fact the most obvious examples of the mixture space would be any convex subset C of $\mathbb{R}$, that is $(C, +, .)$. In what follows we write '+' for $\oplus$ and '.' for $\circ$. It makes life simple and the context should be obvious.[3]

[3]Then the three properties can be written as

(1) $1p + 0q = p$,

(2) $ap + (1-a)q = (1-a)q + ap$, and

(3) $b[ap + (1-a)q] + (1-b)q = (ab)p + (1-ab)q$.

Remark 1. *Note that if for a mixture space the following hold:*

$$ap + (1 - a)p = p \tag{$*$}$$

and

$$b[ap + (1 - a)q] + (1 - b)[cp + (1 - c)q] = [ab + (1 - b)c]p + [b(1 - a) + (1 - b)(1 - c)]q \tag{$\star$}$$

where of course $p, q \in \Pi$, *and* $a, b, c \in [0, 1]$.

Condition ($*$) is straightforward. $ap+(1-a)p = a[1p+0p]+(1-a)p = a[0p+1p]+(1-a)p = 0p+1p = 1p+0p = p$ where the first equality follows from part (1) of the definition, second follows from part (2), third follows from part (3), the fourth follows from part (2) of the definition while the last follows from part (1).

Exercise 1. *Show that ($\star$) holds.*

Let $(\Pi, +, .)$ be a mixture space and $\succsim$ isa binary relation on Π. Consider the following axioms:

Axiom 1′ : $\succsim$ is a preference relation.

Axiom 2′ : For all $p, q, r \in \Pi$, $p \succ q$ and $a \in (0, 1) \implies ap + (1 - a)r \succ aq + (1 - a)r$.

Axiom 3′ : For all $p, q, r \in \Pi$, $p \succ q \succ r \implies \exists\, a, b \in (0, 1)$ such that $ap + (1 - a)r \succ q \succ bp + (1 - b)r$.

As in the vNM model to prove the representation theorem we will need the help of some lemmas. So we shoot the following:

Lemma 3. *If* $\succsim$ *satisfies Axioms* $1' - 3'$, *then:*

(i) $p \succ q$ *and* $0 \le a < b \le 1 \implies bp + (1 - b)q \succ ap + (1 - a)q$

(ii) $p \succsim q \succsim r$ *and* $p \succ r \implies \exists$ *a unique* $a^* \in [0, 1]$ *such that* $q \sim a^*p + (1 - a^*)r$

(iii) $p \sim q$ *and* $a \in [0, 1] \implies ap + (1 - a)r \sim aq + (1 - a)r\ \forall r \in \Pi$.

Proof. □

Theorem 3. *Suppose* Π *is a mixture space and* $\succsim$ *is a binary relation on* Π. *then* $\succsim$ *satisfies Axioms* $1' - 3'$ *iff it has a linear representation.*

Proof. □

The main difference in the proof here will be to get around the problem of best and worst prize that used in Lemma 2.

Remark 2. *Note that in theorem 1, we found a linear representation when X is finite. Now let X be an arbitrary set and $\mathcal{L}^0(X)$ the set of simple lotteries over X. Then, the set $\mathcal{L}^0(X)$ with the usual addition and multiplication forms a mixture space. Thus using the Mixture Space theorem, it is easy to see that the vNM expected utility result can be extended to an arbitrary set and the space of simple lotteries. A binary relation $\succsim$ satisfying our favourite axiom has a linear representation of the form $U : \mathcal{L}^0(X) \to \mathbb{R}$. Moreover, note that in proving proposition 1 we did not use the finiteness of X, only the fact that the support of the lotteries on X is finite. Thus, the same proposition holds for any arbitrary X and $\mathcal{L}^0(X)$. Therefore, we have the following result:*

Theorem 4. *Let X be an arbitrary set and $\mathcal{L}^0(X)$ the set of simple lotteries over X. A binary relation $\succsim$ satisfies Axioms 1-3 iff there exists a function $U : \mathcal{L}^0(X) \to \mathbb{R}$ that represents $\succsim$. Also there exists $u : X \to \mathbb{R}$ such that $U(p) = \sum_{x \in X} p(x)u(x)$ for all $p \in \mathcal{L}^0(X)$.*