Preference, choice function, utility representation

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1 Preference relations

Let X be a finite set. Let R be a binary relation on X, that is, $R \subset X \times X$. We can write $(x, y) \in R$ or simply xRy.

A binary relation R is said to be

1. *complete* if for each $x, y \in X,\ xRy$ or yRx.
2. *transitive* if for each $x, y, z \in X,\ xRy \ and \ yRz \implies xRz$.
3. *asymmetric* if for each $x, y \in X,\ xRy \implies \neg yRx$.[1]
4. *negative transitive* if for each $x, y, z \in X,\ xRy \implies xRz \ or \ zRy$.[2]
5. *anti-symmetric* if for each $x, y \in X,\ xRy \ and \ yRx \implies = y$.
6. *acyclic* if for each $\{x_1, \ldots, x_n\} \subset X,\ x_i R x_{i+1} \forall\ i = 1, \ldots, n-1 \implies \neg x_n R x_1$.

Definition 1. *A binary relation that satisfies completeness and transitivity is called a preference relation.*

Definition 2. *A binary relation that satisfies asymmetry and negative transitivity is called a K-preference relation.*

For any binary relation R, let $\succ_R$ denote the binary relation obtained from R as follows: $x \succ_R y$ iff xRy and $\neg yRx$. Similarly define $\sim_R$ by $x \sim_R y$ iff xRy and yRx.

For any binary relation P, let $\succsim_P$ by $x \succsim_P y$ iff $\neg yPx$.

Proposition 1. *A) R is a preference relation implies $\succ_R$ is a K-preference relation.*

B) P is a K-preference relation iff $\succsim_P$ is a preference relation.

C) R is a preference relation implies $\succsim_{\succsim_R} = R$.

D) P is a K-preference relation implies $\succsim_{\succsim_P} = P$.

[1] $\neg aRb$ means not aRb.

[2] An alternate and equivalent way to define this is the following: R is negative transitive if for each $x, y, z \in X,\ \neg xRy \ and \ \neg yRz \implies \neg xRz$.

2 Choice functions

For any nonempty set X, let $\mathcal{P}(X)$ denote the set of all nonempty subsets of X. A mapping $c : \mathcal{P}(X) \to \mathcal{P}(X)$, is called a choice function if $c(A) \subset A$ for all $A \in \mathcal{P}(X)$. The interpretation is: If the decision maker (DM) is offered a choice of anything in the set A, he says that any member of $c(A)$ will do use fine. Choice functions are the building blocks of revealed preference theory.

Next, let $A, B \in \mathcal{P}(X)$.

Houthakker's Axiom. $c(A) \cap B \neq \emptyset \implies c(B) \cap A \subset c(A)$.
Alternatively we can also write this as: if $x, y \in A$ and B and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$.

Sen's α Axiom: $c(A \cap B) \cap A \subset A$.
Alternatively we can also write this as: if $x \in B \subset A$ and $x \in c(A)$, then $x \in c(B)$. Sen's paraphrase of this is: If the world champion in some game is a Pakistani, then he must be the champion of Pakistan.

Sen's β Axiom. $c(A \cup B) \cap B \neq \emptyset$ implies $c(B) \subset c(A \cup B)$.
Alternatively we can also write this as: if $x, y \in c(A), A \subset B$ and $y \in c(B)$, then $x \in c(B)$. Sen's paraphrase of this is: If the world champion in some game is a Pakistani, then, all champions (in this game) of Pakistan are also world champions.

For any binary relation R on X, define two functions, $c(., R) : \mathcal{P}(X) \to \mathcal{P}(X) \cup \emptyset$ and $c_R : \mathcal{P}(X) \to \mathcal{P}(X) \cup \emptyset$ as follows:

$$c(A, R) = \{x \in A | \nexists y \in A \text{ s.t. } yRx\}$$
$$c_R(A) = \{x \in A | xRy \forall y \in A\}$$

Now we state the two main theorems that characterize choice functions.

Theorem 1. *If X is finite, then R is acyclic iff $c(., R)$ is a choice function.*

Theorem 2. *The choice function c satisfies the Houthakker's Axiom iff it satisfies Sen's Condition α and β iff $\exists$ a preference relation R such that $c = c_R$.*

3 Utility Representation

We say that the function $U : X \to \mathbb{R}$ represents the binary relation R if

$$xRy \text{ iff } U(x) \geq U(y) \; \forall x, y \in X.$$

We say that the function $U : X \to \mathbb{R}$ K-represents the binary relation R of

$$xRy \text{ iff } U(x) > U(y) \; \forall x, y \in X.$$

Proposition 2. *U represents R iff it K-represents $\succ_R$, and U represents $\succsim_P$ iff it K-represents P.*

Next, we state the main characterization of preferences in terms of a utility function.

Theorem 3. *For a finite set X and a binary relation R on X, there exists a function U that represents R iff R is a preference relation.*

We can also prove a similar result when X is countably infinite.

Theorem 4. *Let X be a countably infinite set and binary relation RR on X, there exists a function U that represents R iff R is a preference relation.*